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Probabilistic HyperRough Set and Covering HyperRough Set

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Abstract

Rough set theory provides a mathematical framework for approximating subsets using lower and upper bounds defined by equivalence relations, effectively capturing uncertainty in classification and data analysis. Building on these foundational ideas, further generalizations such as Hyperrough Sets and Superhyperrough Sets have been developed. Probabilistic Rough Sets provide a framework for estimating uncertainty by utilizing membership probabilities, allowing for the definition of lower and upper approximations based on specified threshold values. Covering rough sets approximate information via overlapping covers, providing lower definite and upper possible boundaries when true partitions are unavailable. In this paper, we introduce newly defined concepts of the Probabilistic HyperRough Set and Covering HyperRough Set, as well as the Probabilistic SuperHyperRough Set and Covering SuperHyperRough Set. These models extend the existing frameworks of the Probabilistic Rough Set and Covering Rough Set, respectively.

Keywords: Rough set, Hyperrough set, Covering rough set, Superhyperrough set, Probabilistic rough set.

1|Introduction

Rough set theory, first introduced by Pawlak, has emerged as a fundamental mathematical framework for modeling vagueness and uncertainty in data analysis and decision-making processes [1,3,5]. By employing lower and upper approximations induced by equivalence relations, rough sets allow for the classification of objects when complete information is unavailable. This capability has made rough set theory particularly valuable in areas such as knowledge discovery, pattern recognition, and artificial intelligence, where imprecision and incomplete data are inherent [2,20].

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Over time, numerous extensions of the classical rough set model have been proposed to enhance its expressive power, including probabilistic rough sets, fuzzy rough sets, and multigranular rough sets [17,18,23,35]. Among these, probabilistic rough sets play a crucial role by incorporating membership probabilities, enabling a quantitative interpretation of uncertainty through threshold-based approximations [17,22]. Such probabilistic frameworks have proven effective in three-way decision theory and risk-sensitive decision-making scenarios, where uncertainty must be explicitly controlled rather than merely bounded [18,21].

Despite these advances, classical and probabilistic rough set models are limited in their ability to simultaneously handle complex attribute interactions and higher-order structural relationships. To address this limitation, hyperrough sets and superhyperrough sets were introduced as generalized frameworks that extend rough sets by incorporating multi-attribute mappings and power-setbased attribute combinations [10,13,16]. Hyperrough sets enable a finer granularity of approximation by considering Cartesian products of attribute domains, allowing multiple attributes to jointly influence the approximation process [10–12]. Superhyperrough sets further enhance this flexibility by permitting subsets of attribute values, making them particularly suitable for modeling uncertain, heterogeneous, or partially overlapping information sources [16]. These models have demonstrated potential in applications ranging from project management and cybersecurity to decision analysis under uncertainty [12,14,15]. However, existing hyperrough and superhyperrough frameworks primarily rely on deterministic approximations and equivalence-based structures, which restrict their adaptability in probabilistic environments where uncertainty is inherently stochastic rather than purely relational.

Another important direction in rough set research involves covering-based models, which replace strict partitions with covers composed of overlapping subsets [24-28]. Covering rough sets provide a more realistic representation of real-world information systems, where objects often belong to multiple categories simultaneously and crisp equivalence relations are difficult to justify [25-27].

By defining lower and upper approximations based on set inclusion and intersection within covers, covering rough sets have been successfully applied in knowledge representation, feature selection, and granular computing [26,28]. Recent studies have also explored multi-granulation and neighborhood-based extensions to further enhance flexibility and robustness [24,35,41]. Nevertheless, the integration of covering structures with hyperrough and probabilistic frameworks remains relatively underexplored. Motivated by this gap, the present study introduces probabilistic hyperrough sets and covering hyperrough sets, along with their superhyperrough counterparts, to unify probabilistic reasoning, multi-attribute hyperstructures, and covering-based approximations within a single coherent framework. These proposed models generalize several existing rough set variants and provide a mathematically rigorous foundation for future developments in uncertainty modeling, granular computing, and advanced decision-support systems [17,20,24].

2|Preliminaries

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. Throughout this paper, all sets under consideration are assumed to be finite.

2.1|Rough Set, HyperRough Set, and Superhyperrough Set

A rough set approximates a subset using lower and upper bounds determined by equivalence classes, thereby capturing both certainty and uncertainty in membership [1–3]. The following definitions formalize these concepts.

Definition 1 (Set)**.** [4] A *set* is a well-defined collection of distinct elements or objects. If a is an element of a set A, we write $a \in A$; otherwise, we write $a \notin A$.

Definition 2 (Subset)**.** [4] Let A and B be sets. A is called a *subset* of B, denoted $A \subseteq B$, if every element of A is also an element of B. If $A \subseteq B$ but $A \neq B$, then A is called a *proper subset* of B, denoted $A \subset B$.

Definition 3 (Empty Set)**.** [4] The *empty set*, denoted by $\varnothing$, is the unique set containing no elements. Formally, for any set A, $\varnothing \subseteq A$.

Definition 4 (Universal Set)**.** A *universal set*, denoted by U, is the set that contains all elements under consideration in a particular context. Every set discussed is assumed to be a subset of U.

Definition 5 (Rough Set Approximation)**.** [5] Let X be a nonempty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (also called an indiscernibility relation) on X. The relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for each $x \in X$, where

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ are defined by:

(1) *Lower Approximation*:

$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$

This set contains all elements whose entire equivalence class is contained within U; these elements *definitely* belong to U.

(2) *Upper Approximation*:

$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$

This set contains all elements whose equivalence class has a nonempty intersection with U; these elements *possibly* belong to U.

Thus, the pair $(\underline{U}, \overline{U})$ forms the rough set representation of U, satisfying

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

Example 1 (Weather Forecasting using Rough Sets)**.** Weather Forecasting is the science of predicting atmospheric conditions—like temperature, precipitation, and wind—based on data from satellites, radars, and models ([6–9]). Consider a meteorological dataset collected over 100 days for a specific region. Each day is characterized by four attributes:

- Temperature (in °C),
- Humidity (in %),
- Wind speed (in km/h),
- Cloud cover (in %).

For illustration, assume we have data for five specific days:

Day	Temperature	Humidity	Wind Speed	Cloud Cover
1	20	70	15	80
2	22	75	10	90
3	18	65	20	50
4	21	80	12	85
5	19	68	18	55

Define an equivalence relation on days as follows: two days are equivalent if their temperatures differ by no more than 2°C and their humidities differ by no more than 5%. Suppose this rule yields the equivalence classes

$$[1] = \{1, 2, 4\} \quad \text{and} \quad [3] = \{3, 5\}.$$

Assume historical records indicate that days 1, 2, and 4 were rainy, so we set the target concept $X = \{1, 2, 4\}$. Then the lower approximation $\underline{X}$ is defined as

$$\underline{X} = \{x \in U \mid [x] \subseteq X\}.$$

Since the class [1] is entirely contained in X, days 1, 2, and 4 are in $\underline{X}$; however, the class [3] is not, so days 3 and 5 fall into the boundary region. This analysis helps forecasters handle uncertainties by grouping similar weather days.

The *HyperRough Set* extends rough set theory by incorporating multiple attributes. Its formal definition is given below [10–12].

Definition 6 (HyperRough Set)**.** [10,13] Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Let $R \subseteq X \times X$ be an equivalence relation on X, with $[x]_R$ denoting the equivalence class of x. A *HyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$ a subset $F(a) \subseteq X$.
- For each $a \in J$, the rough set approximations of $F(a)$ are defined as

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

Here, $\underline{F(a)}$ comprises all elements whose equivalence classes are completely contained within $F(a)$, while $\overline{F(a)}$ contains elements whose equivalence classes intersect $F(a)$. Additionally, the following properties hold for all $a \in J$:

- $\underline{F(a)} \subseteq \overline{F(a)}$.
- If $F(a) = \emptyset$, then $\underline{F(a)} = \overline{F(a)} = \emptyset$.
- If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.

Example 2 (Project Management using Hyperrough Sets)**.** Project Management is the systematic process of planning, organizing, executing, and monitoring tasks to achieve specific goals within defined time, budget, and scope constraints (cf. [14, 15]). Imagine a project portfolio with 6 projects characterized by two attributes:

- Duration (in months),
- Risk Level (on a scale from 1 to 5).

Suppose the following data is available:

Project	Duration	Risk Level
P_1	12	2
P_2	18	4
P_3	15	3
P_4	20	5
P_5	10	1
P_6	16	3

Define an equivalence relation where two projects are equivalent if their durations differ by at most 3 months and their risk levels by at most 1. Assume the following equivalence classes result:

$$[P_1] = \{P_1, P_3, P_5\} \quad \text{and} \quad [P_2] = \{P_2, P_4, P_6\}.$$

Let the target concept X be the set of projects completed on schedule, say $X = \{P_1, P_3, P_5\}$. The Hyperrough Set model uses the Cartesian product of the attribute domains (e.g., Duration × Risk Level) to form refined groups. For example, consider the attribute combination $(12, 2)$ with

$$F((12, 2)) = \{P_1, P_5\}.$$

Then the lower approximation of X with respect to this combination includes projects whose equivalence class is completely contained in X. This multi-attribute analysis allows project managers to evaluate projects more comprehensively by considering both duration and risk factors simultaneously.

An n-SuperHyperRough Set generalizes rough sets by using power sets of attribute values to produce nuanced approximations under uncertainty. The definition of n-SuperHyperRough Sets is described as follows.

Definition 7 (n-SuperHyperRough Set)**.** [10, 16] Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote its power set. Define the set of all possible attribute value combinations as

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

Let $R \subseteq X \times X$ be an equivalence relation on X. An *n-SuperHyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$ (with $A_i \subseteq J_i$ for all i) a subset $F(A) \subseteq X$.
- For each $A \in J$, the lower and upper approximations are defined as

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\}.$$

Thus, $\underline{F(A)}$ consists of all elements whose equivalence classes are entirely contained in $F(A)$, and $\overline{F(A)}$ includes those elements whose equivalence classes intersect $F(A)$. The following properties hold for all $A \in J$:

- $\underline{F(A)} \subseteq \overline{F(A)}$.
- If $F(A) = \emptyset$, then $\underline{F(A)} = \overline{F(A)} = \emptyset$.
- If $F(A) = X$, then $\underline{F(A)} = \overline{F(A)} = X$.
- For any $A, B \in J$,

$$\underline{F(A \cap B)} \subseteq \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \supseteq \overline{F(A)} \cup \overline{F(B)}.$$

Example 3 (Project Management using Superhyperrough Sets)**.** Consider a scenario where a company evaluates 4 projects based on cost and quality. The attributes are defined as follows:

- **Cost** with domain $J_1 = \{\text{Low}, \text{Medium}, \text{High}\}$,
- **Quality** with domain $J_2 = \{\text{Poor}, \text{Fair}, \text{Good}\}$.

For a more flexible analysis, the Superhyperrough Set model considers the power sets $\mathcal{P}(J_1)$ and $\mathcal{P}(J_2)$. For instance, an element of $\mathcal{P}(J_1)$ could be $\{\text{Low}\}$ or $\{\text{Low}, \text{Medium}\}$, and similarly for $\mathcal{P}(J_2)$. Define the super attribute space as

$$J^* = \mathcal{P}(J_1) \times \mathcal{P}(J_2).$$

Suppose the following project data is given:

Project	Cost	Quality
P_1	Low	Good
P_2	Medium	Fair
P_3	High	Poor
P_4	Low	Fair

Let the target concept X be projects that are considered "acceptable" based on performance, say $X = \{P_1, P_2, P_4\}$. Using the super attribute space, we can form attribute combinations such as

$$(\{\text{Low}\}, \{\text{Good}, \text{Fair}\}),$$

and define

$$F^*\big((\{\text{Low}\}, \{\text{Good}, \text{Fair}\})\big) = \{P_1, P_4\}.$$

Then, the lower approximation of X with respect to this combination consists of projects for which every project in the corresponding equivalence class is in X. This finer granularity in handling attribute subsets helps decision makers identify projects with acceptable performance more precisely, even when cost and quality assessments are subject to uncertainty.

2.2|Probabilistic Rough Sets

Probabilistic Rough Sets provide a framework for estimating uncertainty by utilizing membership probabilities, allowing for the definition of lower and upper approximations based on specified threshold values [17,18]. These sets offer a flexible and intuitive approach to handling imprecise data. We formally define *Probabilistic Rough Sets* below. Due to their simplicity and effectiveness, they have been extensively studied in numerous research works [19–23].

Definition 8 (Probabilistic Rough Set)**.** (cf. [17, 18]) Let U be a finite universe and let $R \subseteq U \times U$ be an equivalence relation on U. This relation partitions U into disjoint equivalence classes, denoted by $[x]_R$ for each $x \in U$. For any subset $A \subseteq U$, define the *rough membership function* $\mu_A : U \to [0,1]$ by

$$\mu_A(x) = \frac{|A \cap [x]_R|}{|[x]_R|}, \quad \text{for every } x \in U,$$

where $|S|$ denotes the cardinality of set S. Given two thresholds $\alpha, \beta \in [0,1]$ satisfying $0 \leq \beta \leq \alpha \leq 1$, the α-level probabilistic *lower approximation* of A is defined as

$$\underline{A}_\alpha = \{x \in U \mid \mu_A(x) \geq \alpha\},$$

and the β-level probabilistic *upper approximation* of A is defined as

$$\overline{A}_\beta = \{x \in U \mid \mu_A(x) > \beta\}.$$

These definitions yield a probabilistic characterization of the set A: $\underline{A}_\alpha$ comprises those elements that are *almost certainly* in A (with membership at least α), while $\overline{A}_\beta$ contains those elements that *possibly* belong to A (with membership exceeding β).

Example 4 (Example of a Probabilistic Rough Set)**.** Consider the finite universe

$$U = \{1, 2, 3, 4, 5, 6\},$$

and define an equivalence relation R on U such that the equivalence classes are:

$$[1]_R = \{1, 2\}, \quad [3]_R = \{3, 4\}, \quad [5]_R = \{5, 6\}.$$

Let $A \subseteq U$ be given by

$$A = \{1, 2, 3, 5\}.$$

We compute the rough membership function $\mu_A(x)$ for each $x \in U$:

- For $x \in [1]_R = \{1, 2\}$:
$$\mu_A(1) = \frac{|\{1, 2\} \cap A|}{2} = \frac{2}{2} = 1, \quad \mu_A(2) = 1.$$
- For $x \in [3]_R = \{3, 4\}$:
$$\mu_A(3) = \frac{|\{3, 4\} \cap A|}{2} = \frac{1}{2} = 0.5, \quad \mu_A(4) = 0.5.$$
- For $x \in [5]_R = \{5, 6\}$:
$$\mu_A(5) = \frac{|\{5, 6\} \cap A|}{2} = \frac{1}{2} = 0.5, \quad \mu_A(6) = 0.5.$$

Now, select thresholds $\alpha = 0.8$ and $\beta = 0.3$. Then the α-level probabilistic lower approximation of A is

$$\underline{A}_{0.8} = \{x \in U \mid \mu_A(x) \geq 0.8\} = \{1, 2\},$$

since only the elements in the equivalence class $\{1, 2\}$ have a membership value of 1 (which is ≥ 0.8). In contrast, the β-level probabilistic upper approximation of A is

$$\overline{A}_{0.3} = \{x \in U \mid \mu_A(x) > 0.3\} = U,$$

because for every $x \in U$, $\mu_A(x)$ is either 0.5 or 1, both of which exceed 0.3.

Thus, the probabilistic rough set approximations of A are:

$$\underline{A}_{0.8} = \{1, 2\} \quad \text{and} \quad \overline{A}_{0.3} = \{1, 2, 3, 4, 5, 6\}.$$

This example concretely illustrates how the rough membership function and the threshold parameters α and β are used to derive probabilistic approximations of a set.

2.3|Covering Rough Set

In classical rough set theory, an equivalence relation (or partition) is used to approximate a subset of a universe. In many real-world applications, however, the information available does not naturally yield a partition but rather a cover – a family of overlapping subsets whose union is the entire universe. Covering rough sets extend the classical model by using such covers to define approximations [24–28].

Definition 9 (Cover)**.** (cf. [25]) Let U be a finite, nonempty set (called the universe). A collection $\mathcal{C}$ of nonempty subsets of U is called a *cover* of U if

$$\bigcup_{C \in \mathcal{C}} C = U.$$

Unlike a partition, the subsets in $\mathcal{C}$ may overlap; that is, for some $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$, we may have $C_1 \cap C_2 \neq \emptyset$.

Definition 10 (Covering Rough Set)**.** [25] Let U be a finite universe and let $\mathcal{C}$ be a cover of U. For any subset $X \subseteq U$, we define:

(1) The *covering lower approximation* of X with respect to $\mathcal{C}$ as

$$\underline{X}_{\mathcal{C}} = \bigcup \{C \in \mathcal{C} \mid C \subseteq X\}.$$

This is the union of all covering sets that are completely contained in X. Intuitively, these are the elements that *definitely* belong to X as their entire covering set is a subset of X.

(2) The *covering upper approximation* of X with respect to $\mathcal{C}$ as

$$\overline{X}_{\mathcal{C}} = \bigcup \{C \in \mathcal{C} \mid C \cap X \neq \emptyset\}.$$

This is the union of all covering sets that have a nonempty intersection with X. Intuitively, these are the elements that *possibly* belong to X because at least part of their covering set intersects X.

The pair $\left(\underline{X}_{\mathcal{C}}, \overline{X}_{\mathcal{C}}\right)$ is called the *covering rough set* of X with respect to the cover $\mathcal{C}$.

A special case occurs when $\mathcal{C}$ is a partition. In that case, the covering lower and upper approximations coincide with the classical rough set approximations.

Example 5 (Example of a Covering Rough Set)**.** Consider the universe

$$U = \{1, 2, 3, 4, 5, 6\}.$$

Define a cover $\mathcal{C}$ of U by

$$\mathcal{C} = \{C_1, C_2, C_3, C_4\},$$

where

$$C_1 = \{1, 2, 3\}, \quad C_2 = \{2, 3, 4\}, \quad C_3 = \{4, 5\}, \quad C_4 = \{5, 6\}.$$

Note that

$$\bigcup_{i=1}^{4} C_i = \{1, 2, 3, 4, 5, 6\} = U.$$

Now, let

$$X = \{2, 3, 4, 5\} \subseteq U.$$

We compute the approximations as follows:

(1) **Covering Lower Approximation:** We include every $C \in \mathcal{C}$ that is entirely contained in X:

- $C_1 = \{1, 2, 3\}$ is not included because $1 \notin X$.
- $C_2 = \{2, 3, 4\} \subseteq X$ is included.
- $C_3 = \{4, 5\} \subseteq X$ is included.
- $C_4 = \{5, 6\}$ is not included because $6 \notin X$.

Therefore,

$$\underline{X}_{\mathcal{C}} = C_2 \cup C_3 = \{2, 3, 4\} \cup \{4, 5\} = \{2, 3, 4, 5\} = X.$$

(2) **Covering Upper Approximation:** We include every $C \in \mathcal{C}$ that has a nonempty intersection with X:

- $C_1 \cap X = \{2, 3\} \neq \emptyset$, so C_1 is included.
- $C_2 \cap X = \{2, 3, 4\} \neq \emptyset$, so C_2 is included.
- $C_3 \cap X = \{4, 5\} \neq \emptyset$, so C_3 is included.
- $C_4 \cap X = \{5\} \neq \emptyset$, so C_4 is included.

Therefore,

$$\overline{X}_C = C_1 \cup C_2 \cup C_3 \cup C_4 = \{1, 2, 3\} \cup \{2, 3, 4\} \cup \{4, 5\} \cup \{5, 6\} = \{1, 2, 3, 4, 5, 6\} = U.$$

Thus, the covering rough set approximation of X is

$$\left(\underline{X}_C, \overline{X}_C\right) = \left(X, U\right).$$

The *boundary region* of X is given by

$$\mathrm{BND}(X) = \overline{X}_C \setminus \underline{X}_C = U \setminus X = \{1, 6\}.$$

This example illustrates a scenario where X is exactly defined by the cover in the sense that the lower approximation equals X, yet the upper approximation spans the entire universe due to overlapping cover sets.

3|Results of This Paper

This section presents the results obtained in this paper.

3.1|Probabilistic Hyperrough Sets

We define *Probabilistic hyperrough Sets* as follows.

Definition 11 (Probabilistic Hyperrough Set)**.** Let U be a finite universe and let $R \subseteq U \times U$ be an equivalence relation on U that partitions U into disjoint equivalence classes $[x]_R$ for each $x \in U$. Let $T_1, T_2, \ldots, T_m$ be m distinct attributes with corresponding domains $J_1, J_2, \ldots, J_m$. Define the Cartesian product

$$J = J_1 \times J_2 \times \cdots \times J_m.$$

A mapping

$$F : J \to \mathcal{P}(U)$$

assigns to each attribute combination $a = (a_1, a_2, \ldots, a_m) \in J$ a subset $F(a) \subseteq U$. For each $a \in J$, the *rough membership function* of $F(a)$ is defined by

$$\mu_{F(a)}(x) = \frac{|F(a) \cap [x]_R|}{|[x]_R|}, \quad \forall x \in U.$$

Given thresholds $\alpha, \beta \in [0, 1]$ satisfying $0 \leq \beta \leq \alpha \leq 1$, we define the α-level probabilistic *lower approximation* of $F(a)$ by

$$\underline{F(a)}_\alpha = \{x \in U \mid \mu_{F(a)}(x) \geq \alpha\},$$

and the β-level probabilistic *upper approximation* of $F(a)$ by

$$\overline{F(a)}_\beta = \{x \in U \mid \mu_{F(a)}(x) > \beta\}.$$

The pair (F, J), together with the family of approximation operators $\{\underline{F(a)}_\alpha, \overline{F(a)}_\beta \mid a \in J\}$, is called a *Probabilistic Hyperrough Set* over U.

Theorem 1. *A Probabilistic Hyperrough Set generalizes both the classical Probabilistic Rough Set and the Hyperrough Set.*

Proof: We consider two limiting cases:

(1) **Singleton Attribute Space:** If the attribute space J is a singleton (i.e., $J = \{a_0\}$), then the mapping F reduces to a single subset $F(a_0) \subseteq U$. In this case, the rough membership function $\mu_{F(a_0)}(x)$ is identical to that in the standard Probabilistic Rough Set model. Consequently, the approximations $\underline{F(a_0)}_\alpha$ and $\overline{F(a_0)}_\beta$ recover the classical probabilistic rough set approximations.

(2) **Extreme Thresholds:** If we set the thresholds to $\alpha = 1$ and $\beta = 0$, then for every $x \in U$,

$$\underline{F(a)}_1 = \{x \in U \mid \mu_{F(a)}(x) = 1\} \quad \text{and} \quad \overline{F(a)}_0 = \{x \in U \mid \mu_{F(a)}(x) > 0\}.$$

These definitions coincide with those of the Hyperrough Set model, where the lower approximation comprises all elements whose entire equivalence classes are contained in $F(a)$, and the upper approximation includes all elements whose equivalence classes intersect $F(a)$.

Thus, the Probabilistic Hyperrough Set model encompasses both the Probabilistic Rough Set (singleton case) and the Hyperrough Set (extreme threshold case) models. □

Example 6 (Probabilistic Hyperrough Set)**.** Consider the finite universe

$$U = \{1, 2, 3, 4, 5, 6\},$$

with an equivalence relation R defined by the following equivalence classes:

$$[1]_R = \{1, 2\}, \quad [3]_R = \{3, 4\}, \quad [5]_R = \{5, 6\}.$$

Let there be a single attribute T_1 with domain $J_1 = \{\text{red}, \text{blue}\}$. Then,

$$J = J_1 = \{\text{red}, \text{blue}\}.$$

Define the mapping $F : J \to \mathcal{P}(U)$ by:

$$F(\text{red}) = \{1, 2, 3\} \quad \text{and} \quad F(\text{blue}) = \{4, 5, 6\}.$$

For $a = \text{red}$, compute the rough membership function:

- For $x \in [1]_R = \{1, 2\}$:
$$\mu_{F(\text{red})}(x) = \frac{|\{1, 2\} \cap \{1, 2, 3\}|}{2} = \frac{2}{2} = 1.$$
- For $x \in [3]_R = \{3, 4\}$:
$$\mu_{F(\text{red})}(x) = \frac{|\{3, 4\} \cap \{1, 2, 3\}|}{2} = \frac{1}{2} = 0.5.$$
- For $x \in [5]_R = \{5, 6\}$:
$$\mu_{F(\text{red})}(x) = \frac{|\{5, 6\} \cap \{1, 2, 3\}|}{2} = 0.$$

If we choose thresholds $\alpha = 0.8$ and $\beta = 0.3$, then:

$$\underline{F(\text{red})}_{0.8} = \{x \in U \mid \mu_{F(\text{red})}(x) \geq 0.8\} = \{1, 2\},$$

and

$$\overline{F(\text{red})}_{0.3} = \{x \in U \mid \mu_{F(\text{red})}(x) > 0.3\} = \{1, 2, 3, 4\}.$$

Thus, the probabilistic hyperrough approximations for the attribute value "red" are $\underline{F(\text{red})}_{0.8} = \{1, 2\}$ and $\overline{F(\text{red})}_{0.3} = \{1, 2, 3, 4\}$.

Similarly, for $a = \text{blue}$, with $F(\text{blue}) = \{4, 5, 6\}$:

- For $x \in [1]_R = \{1, 2\}$:
$$\mu_{F(\text{blue})}(x) = \frac{|\{1, 2\} \cap \{4, 5, 6\}|}{2} = 0.$$
- For $x \in [3]_R = \{3, 4\}$:
$$\mu_{F(\text{blue})}(x) = \frac{|\{3, 4\} \cap \{4, 5, 6\}|}{2} = \frac{1}{2} = 0.5.$$

- For $x \in [5]_R = \{5,6\}$:

$$\mu_{F(\text{blue})}(x) = \frac{|\{5,6\} \cap \{4,5,6\}|}{2} = \frac{2}{2} = 1.$$

With the same thresholds, we obtain:

$$\underline{F(\text{blue})}_{0.8} = \{x \in U \mid \mu_{F(\text{blue})}(x) \geq 0.8\} = \{5,6\},$$

and

$$\overline{F(\text{blue})}_{0.3} = \{x \in U \mid \mu_{F(\text{blue})}(x) > 0.3\} = \{3,4,5,6\}.$$

This example clearly illustrates how the probabilistic hyperrough set model works.

3.2|Probabilistic n-Superhyperrough Sets

We define *Probabilistic n-Superhyperrough Sets* as follows.

Definition 12 (Probabilistic n-Superhyperrough Set)**.** Let U be a finite universe and $R \subseteq U \times U$ be an equivalence relation on U. Let $T_1, T_2, \ldots, T_n$ be n distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, denote by $\mathcal{P}(J_i)$ its power set, and define the Cartesian product

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

A mapping

$$F : J \to \mathcal{P}(U)$$

assigns to each combination $A = (A_1, A_2, \ldots, A_n) \in J$ (with $A_i \subseteq J_i$) a subset $F(A) \subseteq U$. For each $A \in J$, define the *rough membership function* $\mu_{F(A)} : U \to [0,1]$ by

$$\mu_{F(A)}(x) = \frac{|F(A) \cap [x]_R|}{|[x]_R|}, \quad \forall x \in U.$$

Given thresholds $\alpha, \beta \in [0,1]$ with $0 \leq \beta \leq \alpha \leq 1$, the α-level probabilistic *lower approximation* of $F(A)$ is defined as

$$\underline{F(A)}_{\alpha} = \{x \in U \mid \mu_{F(A)}(x) \geq \alpha\},$$

and the β-level probabilistic *upper approximation* is defined as

$$\overline{F(A)}_{\beta} = \{x \in U \mid \mu_{F(A)}(x) > \beta\}.$$

Then, the pair (F, J) is called a *Probabilistic n-Superhyperrough Set* over U.

Theorem 2. *A Probabilistic n-Superhyperrough Set generalizes both the Probabilistic Hyperrough Set and the classical n-Superhyperrough Set.*

Proof: **(i)** If, for each attribute T_i, the power set $\mathcal{P}(J_i)$ is restricted to singletons (i.e., A_i is a singleton for every i), then J reduces to the Cartesian product $J_1 \times J_2 \times \cdots \times J_n$. In this case, the mapping F becomes identical to that of the Hyperrough Set model, and the approximations $\underline{F(A)}_{\alpha}$ and $\overline{F(A)}_{\beta}$ are the same as those in the Probabilistic Hyperrough Set model.

(ii) Conversely, if the thresholds are set to their extreme values, $\alpha = 1$ and $\beta = 0$, then for each $x \in U$,

$$\underline{F(A)}_{1} = \{x \in U \mid \mu_{F(A)}(x) = 1\} \quad \text{and} \quad \overline{F(A)}_{0} = \{x \in U \mid \mu_{F(A)}(x) > 0\},$$

which coincides with the classical n-Superhyperrough Set definitions.

Thus, the Probabilistic n-Superhyperrough Set model encompasses both the Probabilistic Hyperrough Set and the n-Superhyperrough Set. □

Example 7 (Probabilistic n-Superhyperrough Set Example)**.** Let

$$U = \{1,2,3,4,5\},$$

with an equivalence relation R defined by:

$$[1]_R = \{1,2\}, \quad [3]_R = \{3,4\}, \quad [5]_R = \{5\}.$$

Assume a single attribute T_1 with domain $J_1 = \{A, B\}$. Then, its power set is

$$\mathcal{P}(J_1) = \{\emptyset, \{A\}, \{B\}, \{A, B\}\},$$

and we set $J = \mathcal{P}(J_1)$. Define the mapping $F : J \to \mathcal{P}(U)$ as follows:

$$\begin{aligned} F(\emptyset) &= \emptyset, \\ F(\{A\}) &= \{1, 2, 3\}, \\ F(\{B\}) &= \{4, 5\}, \\ F(\{A, B\}) &= U. \end{aligned}$$

For $A = \{A\}$, we have $F(\{A\}) = \{1, 2, 3\}$. Then:

- For $x \in [1]_R = \{1, 2\}$:
$$\mu_{F(\{A\})}(x) = \frac{|\{1, 2\} \cap \{1, 2, 3\}|}{2} = 1.$$
- For $x \in [3]_R = \{3, 4\}$:
$$\mu_{F(\{A\})}(x) = \frac{|\{3, 4\} \cap \{1, 2, 3\}|}{2} = \frac{1}{2} = 0.5.$$
- For $x \in [5]_R = \{5\}$:
$$\mu_{F(\{A\})}(x) = \frac{|\{5\} \cap \{1, 2, 3\}|}{1} = 0.$$

Choosing thresholds $\alpha = 0.8$ and $\beta = 0.3$, we obtain:

$$\underline{F(\{A\})}_{0.8} = \{x \in U \mid \mu_{F(\{A\})}(x) \geq 0.8\} = \{1, 2\},$$

and

$$\overline{F(\{A\})}_{0.3} = \{x \in U \mid \mu_{F(\{A\})}(x) > 0.3\} = \{1, 2, 3, 4\}.$$

For $A = \{B\}$, with $F(\{B\}) = \{4, 5\}$:

- For $x \in [3]_R = \{3, 4\}$:
$$\mu_{F(\{B\})}(x) = \frac{|\{3, 4\} \cap \{4, 5\}|}{2} = \frac{1}{2} = 0.5.$$
- For $x \in [5]_R = \{5\}$:
$$\mu_{F(\{B\})}(x) = \frac{|\{5\} \cap \{4, 5\}|}{1} = 1.$$
- For $x \in [1]_R = \{1, 2\}$:
$$\mu_{F(\{B\})}(x) = 0.$$

Thus,

$$\underline{F(\{B\})}_{0.8} = \{5\} \quad \text{and} \quad \overline{F(\{B\})}_{0.3} = \{3, 4, 5\}.$$

This example clearly demonstrates how the Probabilistic n-Superhyperrough Set model refines the approximations by incorporating power set combinations of attribute values.

3.3|Covering Hyperrough Set

We define *Covering Hyperrough Set* as follows.

Definition 13 (Covering Hyperrough Set)**.** Let U be a finite universe, C a cover of U, and let $T_1, T_2, \ldots, T_m$ be m distinct attributes with corresponding domains $J_1, J_2, \ldots, J_m$. Define the Cartesian product

$$J = J_1 \times J_2 \times \cdots \times J_m.$$

A mapping

$$F : J \to \mathcal{P}(U)$$

assigns to each attribute combination $a = (a_1, a_2, \ldots, a_m) \in J$ a subset $F(a) \subseteq U$. Then, for each $a \in J$, the *covering hyperrough approximations* of $F(a)$ with respect to the cover $\mathcal{C}$ are defined as:

$$\underline{F(a)}_{\mathcal{C}} = \bigcup \{C \in \mathcal{C} \mid C \subseteq F(a)\},$$

$$\overline{F(a)}_{\mathcal{C}} = \bigcup \{C \in \mathcal{C} \mid C \cap F(a) \neq \emptyset\}.$$

The collection (F, J) together with the family $\{(\underline{F(a)}_{\mathcal{C}}, \overline{F(a)}_{\mathcal{C}}) \mid a \in J\}$ is called a *Covering Hyperrough Set* over U.

Theorem 3. *A Covering Hyperrough Set generalizes both the classical Covering Rough Set and the Hyperrough Set.*

Proof: **Case 1 (Reduction to Covering Rough Set):** If the attribute space J is a singleton (i.e., $J = \{a_0\}$) and $F(a_0) = X$ for some $X \subseteq U$, then the definitions reduce to:

$$\underline{F(a_0)}_{\mathcal{C}} = \underline{X}_{\mathcal{C}} \quad \text{and} \quad \overline{F(a_0)}_{\mathcal{C}} = \overline{X}_{\mathcal{C}},$$

which is exactly the classical covering rough set approximation of X.

Case 2 (Reduction to Hyperrough Set): If the cover $\mathcal{C}$ is a partition of U (i.e., each $C \in \mathcal{C}$ is an equivalence class of an equivalence relation R) and F is constant (so that $F(a) = X$ for all $a \in J$), then the approximations become

$$\underline{F(a)}_{\mathcal{C}} = \{x \in U \mid [x]_R \subseteq X\} \quad \text{and} \quad \overline{F(a)}_{\mathcal{C}} = \{x \in U \mid [x]_R \cap X \neq \emptyset\},$$

which are precisely the hyperrough set approximations of X.

Thus, the Covering Hyperrough Set model indeed generalizes both frameworks. □

Example 8 (Covering Hyperrough Set)**.** Let

$$U = \{1, 2, 3, 4, 5, 6\},$$

and consider the cover

$$\mathcal{C} = \{C_1, C_2, C_3\},$$

with

$$C_1 = \{1, 2\}, \quad C_2 = \{2, 3, 4\}, \quad C_3 = \{5, 6\}.$$

Let there be one attribute T_1 with domain $J_1 = \{\text{red}, \text{blue}\}$ so that

$$J = \{\text{red}, \text{blue}\}.$$

Define the mapping $F : J \to \mathcal{P}(U)$ by:

$$F(\text{red}) = \{1, 2, 3, 4\}, \quad F(\text{blue}) = \{3, 4, 5, 6\}.$$

For $a = \text{red}$, we have $F(\text{red}) = \{1, 2, 3, 4\}$:

- $C_1 = \{1, 2\} \subseteq F(\text{red}) \Rightarrow$ included in the lower approximation.
- $C_2 = \{2, 3, 4\} \subseteq F(\text{red}) \Rightarrow$ included in the lower approximation.
- $C_3 = \{5, 6\}$ does not intersect $F(\text{red}) \Rightarrow$ excluded.

Thus,

$$\underline{F(\text{red})}_{\mathcal{C}} = C_1 \cup C_2 = \{1, 2, 3, 4\}.$$

For the upper approximation, all cover elements that intersect $F(\text{red})$ are included. Here,

$$C_1 \cap F(\text{red}) \neq \emptyset, \quad C_2 \cap F(\text{red}) \neq \emptyset,$$

so

$$\overline{F(\text{red})}_{\mathcal{C}} = C_1 \cup C_2 = \{1, 2, 3, 4\}.$$

For $a = \text{blue}$, with $F(\text{blue}) = \{3, 4, 5, 6\}$:

- $C_1 = \{1, 2\}$ does not intersect $F(\text{blue}) \Rightarrow$ excluded.
- $C_2 = \{2, 3, 4\}$ has nonempty intersection ($\{3, 4\}$) $\Rightarrow$ included in the upper approximation.

- $C_3 = \{5, 6\} \subseteq F(\text{blue}) \Rightarrow$ included in both the lower and upper approximations.

Thus,

$$\underline{F(\text{blue})}_C = C_3 = \{5, 6\},$$
$$\overline{F(\text{blue})}_C = C_2 \cup C_3 = \{2, 3, 4, 5, 6\}.$$

This example demonstrates how the Covering Hyperrough Set model uses both the cover C and the attribute mapping F to yield approximations.

3.4|Covering n-Superhyperrough Set

We define *Covering n-Superhyperrough Set* as follows.

Definition 14 (Covering n-Superhyperrough Set)**.** Let U be a finite universe and let C be a cover of U. Let $T_1, T_2, \ldots, T_n$ be n distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote its power set, and define the Cartesian product

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

A mapping

$$F : J \to \mathcal{P}(U)$$

assigns to each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$ (with $A_i \subseteq J_i$ for each i) a subset $F(A) \subseteq U$. The *covering n-superhyperrough approximations* of $F(A)$ with respect to the cover C are defined as:

$$\underline{F(A)}_C = \bigcup \{C \in C \mid C \subseteq F(A)\},$$
$$\overline{F(A)}_C = \bigcup \{C \in C \mid C \cap F(A) \neq \emptyset\}.$$

The pair (F, J) is then called a *Covering n-Superhyperrough Set* over U.

Theorem 4. *A Covering n-Superhyperrough Set generalizes both the Covering Hyperrough Set and the classical n-Superhyperrough Set.*

Proof: **(i) Reduction to Covering Hyperrough Set:** If for each attribute T_i we restrict $\mathcal{P}(J_i)$ to singleton sets (i.e., every A_i is a singleton), then J becomes

$$J = J_1 \times J_2 \times \cdots \times J_n,$$

and the mapping F coincides with that used in the Covering Hyperrough Set model. Thus, the approximations $\underline{F(A)}_C$ and $\overline{F(A)}_C$ are exactly those of a Covering Hyperrough Set.

(ii) Reduction to Classical n-Superhyperrough Set: If, additionally, the cover C is a partition (i.e., it is induced by an equivalence relation on U), then the approximations reduce to:

$$\underline{F(A)}_C = \{x \in U \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)}_C = \{x \in U \mid [x]_R \cap F(A) \neq \emptyset\},$$

which are precisely the approximations used in the classical n-Superhyperrough Set model.

Therefore, the Covering n-Superhyperrough Set model unifies and generalizes both the Covering Hyperrough Set and the n-Superhyperrough Set. □

Example 9 (Covering n-Superhyperrough Set)**.** Let

$$U = \{1, 2, 3, 4, 5, 6, 7, 8\},$$

and let the cover C of U be given by

$$C = \{C_1, C_2, C_3\},$$

where

$$C_1 = \{1, 2, 3\}, \quad C_2 = \{3, 4, 5\}, \quad C_3 = \{6, 7, 8\}.$$

Assume there is one attribute T_1 with domain $J_1 = \{\text{X}, \text{Y}\}$. Then, the power set is

$$\mathcal{P}(J_1) = \{\emptyset, \{\text{X}\}, \{\text{Y}\}, \{\text{X}, \text{Y}\}\},$$

and we set $J = \mathcal{P}(J_1)$. Define the mapping $F : J \to \mathcal{P}(U)$ by:

$$\begin{aligned} F(\varnothing) &= \varnothing, \\ F(\{\mathrm{X}\}) &= \{1,2,3,4\}, \\ F(\{\mathrm{Y}\}) &= \{4,5,6,7\}, \\ F(\{\mathrm{X},\mathrm{Y}\}) &= U. \end{aligned}$$

For $A = \{\mathrm{X}\}$, $F(\{\mathrm{X}\}) = \{1,2,3,4\}$:

- $C_1 = \{1,2,3\} \subseteq \{1,2,3,4\}$ is included in the lower approximation.
- $C_2 = \{3,4,5\}$ is not completely contained in $\{1,2,3,4\}$ (since $5 \notin F(\{\mathrm{X}\})$), but $C_2 \cap F(\{\mathrm{X}\}) = \{3,4\} \neq \varnothing$ so it is included in the upper approximation.
- $C_3 = \{6,7,8\}$ does not intersect $\{1,2,3,4\}$ and is excluded.

Thus,

$$\underline{F(\{\mathrm{X}\})}_C = C_1 = \{1,2,3\},$$

$$\overline{F(\{\mathrm{X}\})}_C = C_1 \cup C_2 = \{1,2,3,4,5\}.$$

For $A = \{\mathrm{Y}\}$, $F(\{\mathrm{Y}\}) = \{4,5,6,7\}$:

- C_1 does not intersect $F(\{\mathrm{Y}\})$ and is excluded.
- $C_2 = \{3,4,5\}$ has a nonempty intersection with $F(\{\mathrm{Y}\})$ ($\{4,5\}$) and is included.
- $C_3 = \{6,7,8\}$ satisfies $C_3 \cap F(\{\mathrm{Y}\}) = \{6,7\}$ and is also included.

Thus,

$$\underline{F(\{\mathrm{Y}\})}_C = C_3 \cap F(\{\mathrm{Y}\}) \quad (\text{if } C_3 \subseteq F(\{\mathrm{Y}\})), \text{ here } C_3 \nsubseteq F(\{\mathrm{Y}\}),$$

so we take the union of those C completely contained in $F(\{\mathrm{Y}\})$. Suppose only C_3 qualifies partially, then the lower approximation might be empty; however, the upper approximation is given by

$$\overline{F(\{\mathrm{Y}\})}_C = C_2 \cup C_3 = \{3,4,5,6,7,8\}.$$

This example shows that the approximations vary with the chosen attribute combination. The model is flexible enough to capture both disjunctive and conjunctive relationships among attribute values.

4|Conclusion and Future Work

In this paper, we introduced several new variants of Rough Set concepts. Looking ahead, there are multiple possibilities for extending these results by incorporating other related Rough Set frameworks. Specifically, further expansions can be explored by leveraging Fuzzy Rough Sets [29,30], Soft Rough Sets [31,32], Neutrosophic Rough Sets [33,34], Multigranular Rough Sets [35], Tree Rough Sets [36,37], Weighted Rough Sets [38,39], and Neighborhood Rough Sets [40,41]. Each of these frameworks offers a distinct way to manage uncertainty and granularity, and their integration with the newly defined Rough Set variants could yield powerful methods for data analysis and decision-making.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

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